School of Engineering
Brown University

EN2210: Continuum Mechanics

Homework 4: Balance laws, work and energy, virtual work
Due 12:00 noon Friday February 4th

1. Show that the local mass balance equation

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const} + \rho\frac{\partial v_i}{\partial y_i} = 0$$

can be re-written in spatial form as

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \rho v_i}{\partial y_i} = 0$$

Note that

$$\left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{x}=const} = \left.\frac{\partial \rho}{\partial t}\right|_{\mathbf{y}=const} + \frac{\partial \rho}{\partial y_i}v_i$$

and substitute into the first equation. Expanding out the derivative of the product in the second equation shows that the two expressions are equivalent.

2. The stress field

$$\sigma_{ij} = \frac{-3P_k y_k y_i y_j}{4\pi R^5} \qquad R = \sqrt{y_k y_k}$$

represents the stress in an infinite, incompressible linear elastic solid that is subjected to a point force with components P_k acting at the origin (you can visualize a point force as a very large body force which is concentrated in a very small region around the origin).

(a) Verify that the stress field is in static equilibrium

$$\frac{\partial \sigma_{ij}}{\partial y_i} = -\frac{3P_k}{4\pi}\left(\frac{\delta_{ik} y_i y_j}{R^5} + \frac{y_k \delta_{ii} y_j}{R^5} + \frac{y_k y_i \delta_{ij}}{R^5} - 5\frac{y_k y_i y_j}{R^6}\frac{y_i}{R}\right) = 0$$

(except at the origin)

(b) Consider a spherical region of material centered at the origin. This region is subjected to (1) the body force acting at the origin; and (2) a force exerted by the stress field on the outer surface of the sphere. Calculate the resultant force exerted on the outer surface of the sphere by the stress, and show that it is equal in magnitude and opposite in direction to the body force.

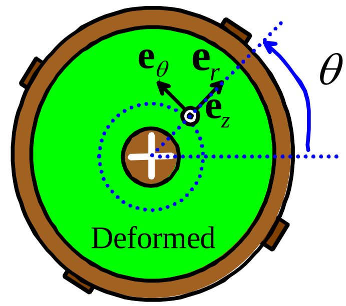
The traction acting on the exterior surface is $\sigma_{ij} n_i \quad n_i = y_i / R$. The resultant force is thus

$$F_j = \frac{-3}{4\pi} P_k \int_S \frac{y_k y_j}{R^3} dA$$

The integral clearly vanishes for $k \neq j$ by symmetry. Choosing $k=j=3$ without loss of generality we can evaluate the remaining integral in spherical-polar coordinates as

$$F_3 = \frac{-3}{4\pi} P_3 \int_0^{2\pi} \int_0^{\pi} \frac{(R\cos\theta)^2}{R^3} R^2 \sin\theta d\theta d\phi \qquad = -P_3$$

3. The figure shows a test designed to measure the viscosity of a fluid. The sample is a hollow cylinder with internal radius a_0 and external radius a_1. The inside diameter is bonded to a fixed rigid cylinder. The external diameter is bonded inside a rigid tube, which is rotated with angular velocity $\omega(t)$. Assume that all material particles in the specimen (green) move circumferentially, with a velocity field (in spatial coordinates) $\mathbf{v} = v_\theta(r,t)\mathbf{e}_\theta$.



(a) Calculate the spatial velocity gradient $\mathbf{L}$ in the basis $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and hence deduce the stretch rate tensor $\mathbf{D}$.

The gradient operator for cylindrical-polar coordinates is (in 2D) $\nabla \equiv \left(\mathbf{e}_r \frac{\partial}{\partial r} + \frac{1}{r}\mathbf{e}_\theta \frac{\partial}{\partial \theta} \right)$. This gives

The velocity gradient follows as $\mathbf{L} = \frac{\partial v_\theta}{\partial r}\mathbf{e}_\theta \otimes \mathbf{e}_r - \frac{v_\theta}{r}\mathbf{e}_r \otimes \mathbf{e}_\theta$.

The stretch rate is the symmetric part of $\mathbf{L}$, i.e. $\mathbf{D} = \frac{1}{2}\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)(\mathbf{e}_\theta \otimes \mathbf{e}_r + \mathbf{e}_r \otimes \mathbf{e}_\theta)$

(b) Calculate the acceleration field

The acceleration is $\left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{y}=const} + \mathbf{L}\cdot\mathbf{v} = \frac{\partial v_\theta}{\partial t}\mathbf{e}_\theta - \frac{v_\theta^2}{r}\mathbf{e}_r$

(c) Suppose that the specimen is homogeneous, has mass density ρ, and may be idealized as a viscous fluid, in which the Kirchhoff stress is related to stretch rate by

$$\boldsymbol{\tau} = 2\mu\mathbf{D} + p(r,t)\mathbf{I}$$

where p is a hydrostatic pressure (to be determined) and μ is the viscosity. Use this to write down an expression for the Cauchy stress tensor in terms of p, expressing your answer as components in $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$

Substituting for the stretch rate, we see that

$$\boldsymbol{\tau} = \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)(\mathbf{e}_\theta \otimes \mathbf{e}_r + \mathbf{e}_r \otimes \mathbf{e}_\theta) + p(\mathbf{e}_r \otimes \mathbf{e}_r + \mathbf{e}_\theta \otimes \mathbf{e}_\theta)$$

(d) Assume steady deformation. Express the equations of equilibrium in terms of $v_\theta(r,t)$.

The equilibrium equation is $\nabla\cdot\boldsymbol{\tau} = 0$, which reduces to

$$\left(\frac{\partial p}{\partial r}\right)\mathbf{e}_r + \left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} - \frac{v_\theta^2}{r^2}\right)\mathbf{e}_\theta = -\frac{v_\theta^2}{r}\mathbf{e}_r$$

(e) Solve the equilibrium equation, together with appropriate boundary conditions, to calculate $v_\theta(r,t)$, and *p(r)*. (The pressure can only be determined to within an arbitrary constant).

The tangential equilibrium equation can be re-written as $\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) = 0$. This equation can easily be integrated to see that $v_\theta = Ar + \frac{B}{r}$. The boundary conditions are $v_\theta = 0 \quad r = a_0 \qquad v_\theta = \omega a_1 \qquad r = a_1$

These two equations can be solved for *A,B*, with the result $v_\theta = \frac{a_1^2 a_0 \omega}{\left(a_1^2 - a_0^2\right)}\left(\frac{r}{a_0} - \frac{a_0}{r}\right)$. (The calculation can also be easily done in maple or mathematica…)

The pressure field can be computed from $\frac{\partial p}{\partial r} = -\frac{v_\theta^2}{r} \Rightarrow p = -\left[\frac{a_1^2 a_0 \omega}{\left(a_1^2 - a_0^2\right)}\right]^2 \left(\frac{r^2}{2a_0^2} - \frac{a_0^2}{2r^2} - 2\log\frac{r}{a_0}\right) + p_0$

Here, p_0 is the pressure at $r = a_0$.

(f) Find an expression for the torque (per unit out of plane distance) necessary to rotate the external cylinder

The torque is $2\pi a_1^2 \tau_{r\theta}(r = a_1) = 2\pi a_1 \mu \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)\Big|_{r=a_1} = \frac{4\pi\mu a_1^2 a_0^2 \omega}{(a_1^2 - a_0^2)}$

(g) Calculate the rate of external work done by the torque acting on the rotating exterior cylinder

The rate of work is $Q\omega = \frac{4\pi\mu a_1^2 a_0^2 \omega^2}{(a_1^2 - a_0^2)}$

(h) Calculate the rate of internal dissipation in the solid as a function of *r*.

The dissipation rate is $\boldsymbol{\tau} : \mathbf{D} = 2\mu \mathbf{D} : \mathbf{D} = \mu\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)^2 = \left[\frac{a_1^2 a_0 \omega}{\left(a_1^2 - a_0^2\right)}\right]^2 \left(\frac{2a_0}{r^2}\right)^2$

(i) Show that the total internal dissipation is equal to the rate of work done by the external moment.

Calculating $\int_{a_0}^{a_1} 2\mu\mathbf{D}:\mathbf{D}2\pi r dr$ gives the same result as (g)…

4. A solid with volume V is subjected to a distribution of traction t_i on its surface. Assume that the solid is in static equilibrium (this requires that t_i exerts no resultant force or moment on the boundary). By considering a virtual velocity of the form $\delta v_i = A_{ij} y_j$, where A_{ij} is a constant tensor, use the principle of virtual work to show that the average stress in a solid can be computed from the shape of the solid and the tractions acting on its surface using the expression

$$\frac{1}{V}\int_V \sigma_{ij} dV = \frac{1}{V}\int_S \frac{1}{2}\left(t_i y_j + t_j y_i\right) dA$$

The virtual work principle gives

$$\int_V \sigma_{ij} A_{ij} dV = \int_S t_i A_{ij} y_j dV$$

Note also that for moment equilibrium

$$\int_S \in_{ijk} y_j t_k dA = 0 \Rightarrow \in_{ipq} \int_S \in_{ijk} y_j t_k dA = \int_S (y_p t_q - y_q t_p) dA = 0$$

Thus

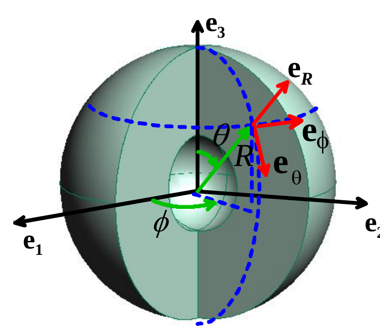
$$\int_S t_i A_{ij} y_j dV = \int_S \frac{1}{2} A_{ij}(t_i y_j + t_j y_i) + \frac{1}{2} A_{ij}(t_i y_j - t_j y_i) dV = \int_S \frac{1}{2} A_{ij}(t_i y_j + t_j y_i) dA$$

Hence, rearranging and noting that A_{ij} is a constant

$$A_{ij}\left(\int_V \sigma_{ij} dV - \int_S \frac{1}{2} A_{ij}(t_i y_j + t_j y_i) dA\right) = 0$$

This must hold for all A_{ij} which shows the required result.

5. The shell shown in the figure is subjected to a radial body force $\mathbf{b} = \rho b(R)\mathbf{e}_R$, and a radial pressure p_a, p_b acting on the surfaces at $R = a$ and $R = b$. The loading induces a spherically symmetric state of stress in the shell, which can be expressed in terms of its components in a spherical-polar coordinate system as $\sigma_{RR}\mathbf{e}_R \otimes \mathbf{e}_R + \sigma_{\theta\theta}\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \sigma_{\phi\phi}\mathbf{e}_\phi \otimes \mathbf{e}_\phi$. By considering a virtual velocity of the form $\delta\mathbf{v} = w(R)\mathbf{e}_R$, show that the stress state is in static equilibrium if



$$\int_a^b \left\{\sigma_{RR}\frac{dw}{dR} + \left(\sigma_{\theta\theta} + \sigma_{\phi\phi}\right)\frac{w}{R}\right\} 4\pi R^2 dR - \int_a^b b(R) w(R) 4\pi R^2 dR - 4\pi a^2 p_a w(a) + 4\pi b^2 p_b w(b) = 0$$

for all $w(R)$. Hence, show that the stress state must satisfy

$$\frac{d\sigma_{RR}}{dR} + \frac{1}{R}\left(2\sigma_{RR} - \sigma_{\theta\theta} - \sigma_{\phi\phi}\right) + b = 0 \qquad \sigma_{RR} = -p_a \ \left(R = a\right) \qquad \sigma_{RR} = -p_b \ \left(R = b\right)$$

For a spherically symmetric state the stretch rate is $\frac{\partial w}{\partial R}\mathbf{e}_R \otimes \mathbf{e}_R + \frac{w}{R}\left(\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi\right)$

The virtual work principle therefore reduces to

$$\int_V \boldsymbol{\sigma} : \delta\mathbf{D}dV - \int_V \rho\mathbf{b}\cdot\delta\mathbf{v}dV - \int_S \mathbf{t}^*\cdot\delta\mathbf{v}dA =$$

$$\int_a^b \left\{\sigma_{RR}\frac{dw}{dR} + \left(\sigma_{\theta\theta} + \sigma_{\phi\phi}\right)\frac{w}{R}\right\}4\pi R^2 dR - \int_a^b b(R)w(R)4\pi R^2 dR - 4\pi a^2 p_a w(a) + 4\pi b^2 p_b w(b) = 0$$

Integrating the first term by parts gives

$$\int_a^b \left\{\frac{1}{R^2}\frac{d(R^2\sigma_{RR})}{dR} + \left(\sigma_{\theta\theta} + \sigma_{\phi\phi}\right)\frac{1}{R} - b(R)\right\} w4\pi R^2 dR$$
$$-4\pi a^2(\sigma_{RR}(a) + p_a)w(a) + 4\pi b^2(\sigma_{RR}(b) + p_b)w(b) = 0$$

This must vanish for all *w*, which gives the required solution.

6. An ideal gas with mass density ρ, pressure *p* and temperature θ has specific internal energy, specific Helmholtz free energy, and stress given by

$$\varepsilon = c_v\theta = \frac{p}{(\gamma-1)\rho} \qquad \psi = c_v\theta - \theta\left(c_v\log\theta - R\log\rho - s_0\right) \qquad \sigma_{ij} = -p\delta_{ij} = -\rho R\theta\delta_{ij}$$

where *R* is the gas constant, c_v is the specific heat capacity (a positive constant), and s_0 is an arbitrary constant. Ideal gases are also characterized by the specific heat at constant pressure $c_p = c_v + R$ and the ratio $\gamma = c_p / c_v$. In addition, heat conduction through an ideal gas is often modeled using Fourier's law

$$q_i = -\kappa\frac{\partial\theta}{\partial y_i}$$

where κ is the thermal conductivity (a positive constant). Show that this constitutive model obeys the free energy imbalance

$$\sigma_{ij}D_{ij} - \frac{1}{\theta}q_i\frac{\partial\theta}{\partial y_i} - \rho\left(\frac{\partial\psi}{\partial t} + s\frac{\partial\theta}{\partial t}\right) \geq 0$$

Consider the first term $\sigma_{ij}D_{ij} = -\rho R\theta\frac{\partial v_i}{\partial y_i} = R\theta\frac{\partial\rho}{\partial t}$ from mass conservation

Consider the second term, $\frac{1}{\theta}q_i\frac{\partial\theta}{\partial y_i} = \frac{\kappa}{\theta}\frac{\partial\theta}{\partial y_i}\frac{\partial\theta}{\partial y_i}$, which is non-negative.

Finally recall that $\psi = \varepsilon - \theta s \Rightarrow s = \frac{\varepsilon - \psi}{\theta} = c_v\log\theta - R\log\rho - s_0$, and note that

$$\frac{\partial\psi}{\partial t} = -c_v\frac{\partial\theta}{\partial t}\log\theta + \frac{\partial\theta}{\partial t}R\log\rho + R\theta\frac{1}{\rho}\frac{\partial\rho}{\partial t} + s_0\frac{\partial\theta}{\partial t}$$

Combining, $\frac{\partial \psi}{\partial t} + s\frac{\partial \theta}{\partial t} = R\theta \frac{1}{\rho}\frac{\partial \rho}{\partial t}$. Hence

$$\sigma_{ij} D_{ij} - \frac{1}{\theta} q_i \frac{\partial \theta}{\partial y_i} - \rho\left(\frac{\partial \psi}{\partial t} + s\frac{\partial \theta}{\partial t}\right) = \frac{\kappa}{\theta}\frac{\partial \theta}{\partial y_i}\frac{\partial \theta}{\partial y_i} \geq 0$$